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Bosonic string theory in Lorentzian principal circle bundles over anti-de-Sitter space AdS_3

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Abstract

In this note we show a bosonic conformal string theory on $U(1)$-bundles over AdS_3. To this end, we first look for r-elastic helices in the space AdS_3, because they generate solutions of the motion equation on backgrounds P which are principal circle-bundles over AdS_3 endowed with the standard metric or generalized Kaluza–Klein metrics. In fact, we reduce the search of $U(1)$-symmetric string configurations on P to the search of r-elastic curves in the orbit space AdS_3. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The anti-de-Sitter (AdS) space emerges in string theory with an increased interest. In the last years, remarkable progress in superstring theory has provided new tools to explore several important issues in understanding the quantum nature of gravity. For instance, one of the most interesting conjectures is in the context of AdS gravity and conformal field theory (CFT) correspondence [12]. It states that string theory in AdS background space–time is equivalent to a supersymmetric gauge theory which lives in the boundary of the space–time.

In this note, we show wide families of r-elastic curves in AdS_3. This will enable us (via the Kaluza–Klein mechanism), to obtain the bosonic conformal string theory associated

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with the Nambu–Goto–Poliakov (NGP) action $\mathcal{S} = \mathcal{W} + \lambda \int \mathrm{d}A$ in Lorentzian principal $U(1)$-bundles over AdS_3 [15]. Here $\mathcal{W}$ denotes the Willmore functional defined on surfaces $\mathcal{W} = \int (H^2 + R)\,\mathrm{d}A$, where H and R are the mean curvature vector field of the surface and the sectional curvature of the total space along the surface, respectively [16].

To begin with, let (M, g) be a semi-Riemannian manifold. For any natural number r, define on the space of inmersed curves γ in M the action,

$$\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2 + \lambda)^{(r+1)/2},$$

where κ is the geodesic curvature of γ, and λ is a real number (here λ can be regarded as a constraint on the length of γ or a coupling constant). The variational problems associated with these actions were considered in [5,6]. The critical points of $\mathcal{F}^r$ are called the *r-elastic curves* or *elasticae* of (M, g), and the Euler–Lagrange equations characterizing these curves were computed there.

Let $\mathbb{C}^2_1$ be the two-dimensional complex linear space endowed with the Hermitian metric $(z, w) = -z_1\bar{w}_1 + z_2\bar{w}_2$, where $z = (z_1, z_2)$, $w = (w_1, w_2) \in \mathbb{C}^2$. The four-dimensional pseudo-Euclidean space $\mathbb{R}^4_2$ can be identified with $\mathbb{C}^2_1$ endowed with the standard pseudo-Euclidean metric $\langle\, , \rangle = \mathrm{Re}(\, ,)$. The AdS space AdS_3 (also denoted by $\mathbb{H}^3_1$ in differential geometry) is defined as the hypersurface $\mathrm{AdS}_3 = \{x \in \mathbb{R}^4_2 : \langle x, x\rangle = -1\}$. The induced metric $\bar{g}$ gives to AdS_3 a Lorentzian structure as a three-dimensional manifold of constant sectional curvature -1. The unit circle $\mathbb{S}^1$ regarded as $U(1)$, acts naturally on AdS_3 to give the standard hyperbolic plane $(\mathbb{H}^2, g_1)$ of constant curvature -4 as orbits space. The canonical projection

$$\pi_1 : \mathrm{AdS}_3 \to \mathbb{H}^2$$

is a semi-Riemannian submersion with time-like closed geodesic fibres.

On the other hand, let H^1 be the unit circle in the Minkowski plane $\mathbb{R}^2_1$. H^1 is the component of the set $\{(x_1, x_2) \in \mathbb{R}^2_1 : -x_1^2 + x_2^2 = -1\}$ containing $(1, 0)$. The group H^1 acts naturally on AdS_3 to give the standard (AdS_2, g_2) of constant curvature -4 as orbits space. The projection

$$\pi_2 : \mathrm{AdS}_3 \to \mathrm{AdS}_2$$

is also a semi-Riemannian submersion with space-like open geodesic fibres.

Geodesics are free (i.e. $\lambda = 0$) trivial (i.e. $\kappa \equiv 0$) r-elastic curves of AdS_3 for any r. In particular, all the fibres of the above natural Hopf fibrations π_1 and π_2 are r-elastic curves.

Recently, Barros [1] classified all the elastic particles that propagate with constant curvature κ in AdS_n for arbitrary dimension n. In fact, from the field equations associated to the elastic energy functional $\mathcal{F}^r$, it is proved that any elastic particle in AdS_n must propagate fully in some AdS_3 totally geodesic in AdS_n. Furthermore, the world-line solutions also have constant torsion τ, and so they are helices in AdS_3. The Euler–Lagrange equation for helices γ in AdS_3 with curvature $\kappa > 0$ and torsion $\tau \neq 0$ to be critical points of $\mathcal{F}^r$ is

$$(r-1)\kappa^2 + r\tau^2 - r = 0.$$

But this is the equation of an ellipse in the (κ, τ)-plane, and hence it provides a one-parameter family of elastic helices, one for each point of the ellipse.

On the other hand, if β is a curve in $\mathbb{H}^2$, the complete lift $\pi_1^{-1}(\beta)$ is a Lorentzian flat surface known as the *Hopf tube* over β in AdS_3 (or a *Hopf torus* if β is a closed curve). Also, the complete lift $\pi_2^{-1}(\beta)$ of a curve β in AdS_2 is called a *B-scroll* [8], and it is a flat Riemannian or Lorentzian surface according to the causal character of β in AdS_2. But we know from [4] that any curve in AdS_3 is a helix if and only if it is a geodesic of either a Lorentzian Hopf tube over a curve with constant curvature in $\mathbb{H}^2$ or a B-scroll over a curve with constant curvature in AdS_2.

All these results can be summed up to give wide families of r-elastic helices in the standard AdS_3. In addition, the complete moduli space of r-elasticae with constant curvature is obtained.

2. Elasticity of fibres in non-standard AdS_3

As we have seen, all the fibres of the Hopf maps π_s, $s = 1, 2$, are geodesics in the space AdS_3 endowed with the standard metric $\bar{g}$, and hence they are (trivial) r-elasticae. Now, $\pi_1 : (\mathrm{AdS}_3, \bar{g}) \to (\mathbb{H}^2, g_1)$, $\pi_2 : (\mathrm{AdS}_3, \bar{g}) \to (\mathrm{AdS}_2, g_2)$ are principal fibre $\mathbb{S}^1$-bundle and H^1-bundle, respectively, which admit canonical principal connections ω_s, $s = 1, 2$, respectively, with non-trivial holonomy. For every positive smooth function f on $\mathbb{H}^2$ or AdS_2, we construct on AdS_3 the *generalized Kaluza–Klein metric*

$$\bar{g}^f = \pi_s^*(g_s) + \varepsilon_s f^2 \omega_s^*(\mathrm{d}\sigma_s^2),$$

where $\varepsilon_1 = -1$, $\varepsilon_2 = 1$, and $\mathrm{d}\sigma_1^2$, $\mathrm{d}\sigma_2^2$ are the standard metrics on $\mathbb{S}^1$ or H^1, respectively, and we write f instead of $f \cdot \pi_s$. It is not difficult to see that

$$\pi_1 : (\mathrm{AdS}_3, \bar{g}^f) \to (\mathbb{H}^2, g_1), \qquad \pi_2 : (\mathrm{AdS}_3, \bar{g}^f) \to (\mathrm{AdS}_2, g_2)$$

are also Riemannian submersions. Then we are interested in the following natural problem:

Characterize those positive smooth functions f on $\mathbb{H}^2$ or AdS_2, *such that all the fibres of $\pi_1 : (\mathrm{AdS}_3, \bar{g}^f) \to (\mathbb{H}^2, g)$ or $\pi_2 : (\mathrm{AdS}_3, \bar{g}^f) \to (\mathrm{AdS}_2, g_2)$ are* (*non-trivial, i.e. non-geodesics*) *r-elastic curves*.

To begin with, we first consider the case $s = 1$, and define in AdS_3 the time-like vector field $V_z = \mathrm{i}z$, $\mathrm{i} = \sqrt{-1}$. Then, V is a vertical vector field and $T = (1/f)V$ is a unit time-like vector field in $(\mathrm{AdS}_3, \bar{g}^f)$ tangent to the fibres. Then, a standard computation involving some well known facts from the theory of semi-Riemannian submersions (see [13]) allows us to obtain,

$$\bar{\nabla}^f_{\bar{X}} \bar{Y} = \overline{\nabla_X Y} + \bar{g}^f(\mathrm{i}\bar{X}, \bar{Y})V, \tag{1}$$

$$\bar{\nabla}^f_{\bar{X}} V = \bar{\nabla}^f_V \bar{X} = \mathrm{i}\bar{X}, \tag{2}$$

$$\bar{\nabla}^f_T T = \frac{\mathrm{grad}(f)}{f} = \mathrm{grad}(\log f), \tag{3}$$

where $\bar{\nabla}^f$ and grad stand for the Levi-Civita connection and the gradient of $\bar{g}^f$, respectively, ∇ is the Levi-Civita connection of $(\mathbb{H}^2, g_1)$ and overbars on vector fields mean lifted objects.

Notice that from the last equation, critical points of f produce geodesic fibres, and all the fibres of π_1 are geodesics if and only if f is a constant.

In order to compute the curvature and torsion of a fibre, write down its Frenet equations,

$$\bar{\nabla}^f_T T = \kappa N, \tag{4}$$

$$\bar{\nabla}^f_T N = \kappa T + \tau B, \tag{5}$$

$$\bar{\nabla}^f_T B = -\tau N, \tag{6}$$

where N and B are the normal and binormal vectors of the fibre, respectively. From (3) and (4) we have,

$$\kappa = \frac{\|\mathrm{grad}(f)\|}{f}, \qquad N = \frac{\mathrm{grad}(f)}{\|\mathrm{grad}(f)\|}. \tag{7}$$

This means that the fibres have constant curvature. As N is a horizontal vector, from (2) and (6) we have,

$$\tau B = \bar{\nabla}^f_N T. \tag{8}$$

By using a local parametrization of the Hopf tube $\pi_1^{-1}(\beta)$, and with some similar arguments to that of [3], it is not difficult to see that $\tau = f$. Then, we have obtained the following:

For any positive smooth function f on the hyperbolic plane $\mathbb{H}^2$, the fibre $\gamma_p = \pi_1^{-1}(p)$ on a point $p \in \mathbb{H}^2$, which is not a critical point of f is a helix with curvature $\kappa = \|\mathrm{grad}(f)(p)\|/f(p)$ and torsion $\tau = f(p)$.

Now, the Euler–Lagrange equation of the action $\mathcal{F}^r$ is

$$(r+1)(\kappa^2+\lambda)^{(r-1)/2}\bar{R}^f(\bar{\nabla}^f_T T, T)T - \bar{\nabla}^f_T[(\kappa^2+\lambda)^{(r-1)/2}((2r+1)\kappa^2-\lambda)T]$$
$$+(r+1)\bar{\nabla}^f_T\,\bar{\nabla}^f_T[(\kappa^2+\lambda)^{(r-1)/2}\bar{\nabla}^f_T T] = 0,$$

where $\bar{R}^f$ is the curvature tensor of $(\mathrm{AdS}_3, \bar{g}^f)$. If we restrict the attention to free elastic fibres and use the above Frenet equations, the motion equation reduces to

$$\kappa^r[(r+1)\bar{R}^f(N,T)T - (r\kappa^2 + (r+1)\tau^2)N] = 0. \tag{9}$$

From this equation we see that on critical points of f, the fibres are geodesics, and so, r-elastic curves for any r. Now, we suppose that f is free of critical points (i.e. $\kappa \neq 0$). The term on the curvature tensor is

$$\bar{R}^f(N,T)T = \left(N(\kappa) + \tau^2 + \frac{N(f)}{f}\kappa\right) + \kappa\,\bar{\nabla}^f_N N.$$

Since N and $\bar{\nabla}^f_N N$ are orthogonal vector fields, Eq. (9) splits up into the following:

$$\bar{\nabla}^f_N N = 0, \tag{10}$$

$$(r+1)N(\kappa) + (r+1)\frac{N(f)}{f}\kappa - r\kappa^2 = 0. \tag{11}$$

We only have to substitute $\kappa = N(f)/f$ in the last equation, to give

$$(r+1)fN(N(f)) - r(N(f))^2 = 0. \tag{12}$$

With the appropriate changes, all the above considerations can be repeated for the second Hopf map $\pi_2 : \mathrm{AdS}_3 \to \mathrm{AdS}_2$. Indeed, the curvature, torsion, and normal vector of a fibre are given by

$$\kappa = -\varepsilon_2 \frac{\|\mathrm{grad}(f)\|}{f}, \qquad \tau = f, \qquad N = \frac{\mathrm{grad}(f)}{\|\mathrm{grad}(f)\|},$$

respectively, and $\varepsilon_2 = 1$ or -1 according whether $\mathrm{grad}(f)$ is a space-like or time-like vector field on AdS_2. Consequently, we have obtained the following:

For any positive smooth function f on AdS_2, *the fibre $\gamma_p = \pi_2^{-1}(p)$ on a point $p \in \mathrm{AdS}_2$, which is not a critical point of f is a helix with curvature $\kappa = -\varepsilon_2(\|\mathrm{grad}(f)(p)\|/f(p))$ and torsion $\tau = f(p)$.*

It is not difficult to see that with suitable first order boundary data, the Euler–Lagrange equation of the $\mathcal{F}^r$-dynamic is,

$$(r+1)(\varepsilon_2\kappa^2+\lambda)^{(r-1)/2}\bar{R}^f(\bar{\nabla}^f_T T, T)T + \bar{\nabla}^f_T[(\varepsilon_2\kappa^2+\lambda)^{(r-1)/2}((2r+1)\varepsilon_2\kappa^2 - \lambda)T]$$
$$+(r+1)\bar{\nabla}^f_T\, \bar{\nabla}^f_T[(\varepsilon_2\kappa^2+\lambda)^{(r-1)/2}\, \bar{\nabla}^f_T T] = 0.$$

Some substitutions and considerations similar to the π_1-case on this equation allow us to get just the same Eqs. (10) and (12). Therefore, we state the following result:

Let f a positive smooth function on $\mathbb{H}^2$ or AdS_2. Then, all the fibres of π_s, $s = 1, 2$, are r-elastic curves in $(\mathrm{AdS}_3, \bar{g}^f)$ if and only if

1. *the unitary field* $N = \mathrm{grad}(f)/\|\mathrm{grad}(f)\|$ *defines a unit speed geodesic flow on* AdS_3,
2. *along this N-flow, f evolves according to*

$$(r+1)fN(N(f)) - r(N(f))^2 = 0.$$

3. Bosonic string theory in circle bundles over AdS_3

It is known that the first fundamental group of AdS_3 is isomorphic to the group $\mathbb{Z}$. Then, for each monomorphism $\phi : \mathbb{Z} \to \mathbb{S}^1$ we can construct [9] a principal circle bundle

$$p : P \to \mathrm{AdS}_3,$$

with flat connection θ. Now, for a given metric h in AdS_3, we consider the Kaluza–Klein metric in P, $\bar{h} = p^*(h) + \theta^*(\mathrm{d}t^2)$, where $\mathrm{d}t^2$ is the standard metric in the unit circle. The map p becomes then a Riemannian submersion.

In the Lorentz space $(P, \bar{h})$ we have the bosonic conformal string theory associated to the combined NGP action [15],

$$S = \mathcal{W} + \lambda \int \mathrm{d}A = \int (H^2 + R + \lambda)\, \mathrm{d}A,$$

where $\mathcal{W} = \int (H^2 + R)\,\mathrm{d}A$ is the Willmore action on surfaces defined in Section 1. This Willmore action is invariant under conformal changes of the metric $\bar{h}$ in P [16], and it was generalized to n-dimensional submanifolds [7]. The critical points of $\mathcal{W}$ are the *Willmore surfaces* in $(P, \bar{h})$. When $(P, \bar{h})$ is flat, the action $\mathcal{S}$ is given by $\mathcal{S} = \int (H^2 + \lambda)$, and it coincides with the classical NGP action [10,11,15].

On the other hand, in order to obtain all possible $U(1)$-invariant configurations of NGP string theory in $(P, \bar{h})$, we first characterize those surfaces in $(P, \bar{h})$ which are $U(1)$-invariant. It is not difficult to see that given a curve γ in the orbit space AdS_3, then $p^{-1}(\gamma)$ is a $U(1)$-invariant surface in $(P, \bar{h})$. The converse of this assertion also holds, i.e. given a $U(1)$-invariant surface M in P, there exists an immersed curve γ in the orbit space AdS_3 such that $M = p^{-1}(\gamma)$. Besides, M is an embedded surface in P if and only if the curve γ has no self-intersections. As a consequence, the space of $U(1)$-invariant surfaces of $(P, \bar{h})$ can be identified with the manifold of curves in AdS_3.

Now, we can obtain string configurations associated to the action $\mathcal{S}$ which have certain degree of symmetry. To be more precise, the symmetry induced from the gauge group $U(1)$. At this point we recall an useful argument due to Palais [14] known as the *principle of symmetric criticality*. This principle is the key argument to exploit the existence of symmetries (if they exist) to solve problems in mathematics and physics. For the sake of simplicity we resume the formulation of this principle in the following way: *the critical points of the action $\mathcal{S}$ defined on the whole space Ω of surfaces which are $U(1)$-symmetric, are the critical points of $\mathcal{S}$ but restricted to the submanifold $\Omega_{U(1)}$ of $U(1)$-symmetric surfaces*. Since we are assuming that $U(1)$ preserves the action $\mathcal{S}$, we apply the principle of symmetric criticality. To obtain that $M = p^{-1}(\gamma)$ is a solution of the Euler–Lagrange equation associated to $\mathcal{S}$ defined on Ω, it is enough to take variations of M in $\Omega_{U(1)}$.

To compute this restriction, we can apply a result of [2] to see that the mean curvature H of $M = p^{-1}(\gamma)$ in $(P, \bar{h})$ and the curvature κ of γ in (AdS_3, h) are related by

$$H^2 = \tfrac{1}{4}k^2. \tag{13}$$

As $R = 0$, if we put (13) in the action $\mathcal{S}$ defined in $\Omega_{U(1)}$, we see that it is a multiple of the action

$$\mathcal{F}^1 = \int (k^2 + \lambda)\,\mathrm{d}s,$$

defined on the space of curves inmersed in the orbit space (AdS_3, h).

The above considerations, allow us to state the following result:

An $U(1)$-invariant world-sheet $M = p^{-1}(\gamma)$ is a configuration of NGP string theory in $(P, \bar{h})$, if and only if γ is a 1-elastic curve in AdS_3, i.e. a critical point of the elastic-energy action $\mathcal{F}^1$.

This result can be considered as a reduction of symmetry method for string configurations. It can be also interpreted as a kind of holographic principle, which relate symmetric soliton configurations of bosonic string theories to classical particles that evolve along r-elastic world-lines in orbit spaces (AdS_3) obtained when reduce the gauge symmetry.

As an application of the results obtained in Sections 1 and 2, we can exhibit a wide family of specific soliton solutions to the string theory in $(P, \bar{h})$, by taking either

1. $\bar{h} = p^*(\bar{g}) + \theta^*(\mathrm{d}t^2)$, or
2. $\bar{h} = p^*(\bar{g}^f) + \theta^*(\mathrm{d}t^2)$,

where $(\bar{g}^f)$ is the generalized Kaluza–Klein metric on AdS_3 over $\mathbb{H}^2$ or AdS_2.

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